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Short Communication

Vibration of circular membranes with linearly varying density along a diameter

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1. Introduction

The vibration analysis of composite circular and annular membranes has been studied in recent years with results obtained using exact solutions, energy methods and finite element analysis. The composite circular membrane is usually described in terms of variable density as a function of the radius of the membrane. In this study the density is assumed to vary linearly along the diameter of the membrane and it follows that density variation is nonaxisymmetric but there is one axis of symmetry. For a complete vibration analysis the entire membrane must be modeled. The finite element method is used and the element is formulated in polar coordinates since the variation in density is described using polar coordinates. Results are given for frequency of vibration and basic mode shapes are illustrated.

2. Governing equations

The method used by Strock and Yu [1] to describe the linear variation of thickness for a circular disk will be used to describe the variation in density for a circular membrane. A dimensionless parameter R that varies between zero and one is defined based

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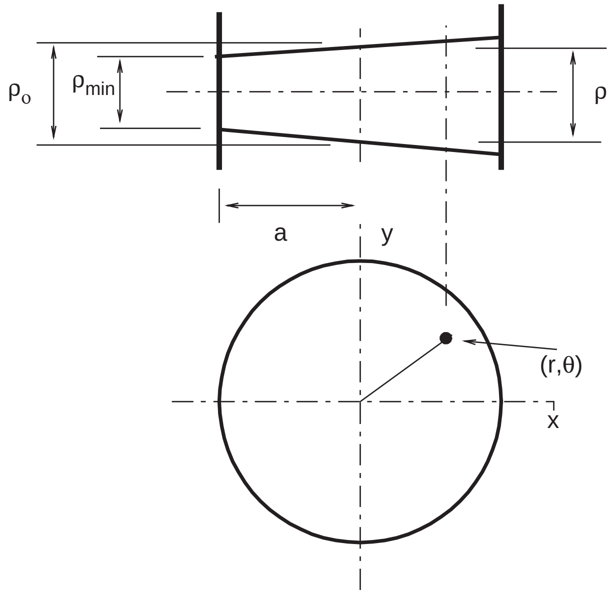

Fig. 1. Variable density circular membrane.

upon Fig. 1 as

$$R = 1 - \rho_{\min}/\rho_0, \tag{1}$$

where ρ_0 is the density at the midpoint of the membrane. When $R=0$ the density is constant ρ_0 for the entire membrane. When $R=1$ the density becomes zero at $x=-a$ and $2\rho_0$ at $x=a$. The density is constant ρ_0 along the y-axis for all values of R. In terms of R and the polar coordinates (r,θ) the variable density is described following Ref. [1] as

$$\rho = \rho_0[1 + R(r/a)\cos\theta]. \tag{2}$$

The governing equation for membrane vibration in (r,θ) coordinates is

$$\nabla^2 w = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = \frac{\rho}{S}\frac{\partial^2 w}{\partial t^2}, \tag{3}$$

where $w = w(r,\theta,t)$ is deflection, S is the constant membrane tension and ρ is the membrane density defined by Eq. (2). Assume motion that can be described with circular frequency ω,

$$w(r,\theta,t) = W(r,\theta)\mathrm{e}^{\mathrm{i}\omega t}. \tag{4}$$

An approximate solution can be formulated in terms of the functional

$$J(W) = \int\int_A \left\{ S\left[\left(\frac{\partial W}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial W}{\partial \theta}\right)^2\right] - \rho\omega^2 W^2 \right\} r\,\mathrm{d}r\,\mathrm{d}\theta. \tag{5}$$

In this application the finite element method is used to obtain a solution for Eq. (3) via the functional Eq. (5).

3. Membrane finite element in polar coordinates

A nine node Lagrangian element is used to model the circular area of the membrane. The shape functions are the same as those discussed by Buchanan and Peddieson [2] for modeling the cross-section of a circular cylinder. The deflection of the membrane is assumed as

$$W = [N]\{W\}, \tag{6}$$

where $[N]$ are the shape functions and $\{W\}$ are the nodal point unknowns. Eq. (5) leads to a governing finite element equation that can be written as

$$[K]\{W\} - \omega^2[M]\{W\} = 0, \tag{7}$$

where

$$[K] = \int_A [B]^{\mathrm{T}}[S][B]\,\mathrm{d}A, \quad [M] = \int_A [N]^{\mathrm{T}}[\rho][N]\,\mathrm{d}A. \tag{8}$$

The $[B]$ matrix is defined in terms of an operator matrix $[L]$ and the shape function matrix. Eq. (5) is used to define $[B]$ as

$$[B] = [L][N] = \begin{bmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial}{\partial \theta} \end{bmatrix} [N_1\ N_2 \cdots N_9]. \tag{9}$$

Table 1
Frequencies $\Omega = \omega a\sqrt{\rho_0/S}$ for circular membranes with variable density

Mode	R						
	0[a]		0.0	0.25	0.50	0.75	1.0
1	2.404	(0)	2.405	2.340	2.385	2.361	2.331
2	3.832	(1)	3.832	3.820	3.785	3.732	3.668
3	3.832	(1)	3.832	3.834	3.839	3.843	3.840
4	5.135	(2)	5.136	5.132	5.116	5.079	5.017
5	5.135	(2)	5.136	5.132	5.119	5.091	5.048
6	5.520	(0)	5.520	5.513	5.497	5.484	5.479
7	6.379	(3)	6.381	6.379	6.370	6.345	6.296
8	6.379	(3)	6.381	6.379	6.370	6.349	6.309
9	7.016	(1)	7.016	6.994	6.933	6.847	6.747
10	7.016	(1)	7.016	7.021	7.036	7.062	7.090
11	7.586	(4)	7.592	7.593	7.591	7.578	7.544
12	7.586	(4)	7.592	7.593	7.591	7.580	7.553
13	8.417	(2)	8.419	8.408	8.361	8.261	8.126
14	8.417	(2)	8.419	8.410	8.383	8.338	8.272
15	8.654	(0)	8.656	8.647	8.642	8.668	8.685

Numbers in parentheses correspond to the circular wave number for $R=0.0$ taken from Ref. [3].
[a]Exact frequencies from Ref. [3].

4. Frequency and mode shapes

Results for frequency of vibration have been computed for both circular and annular membranes. The radius defining the interior boundary of an annular membrane in defined as b. The nondimensional frequency Ω is defined as

$$\Omega = \omega a\sqrt{\rho_0/S}, \tag{10}$$

where the outside radius of the membrane a, ρ_0 and S are assumed to have unit values. It follows that the nondimensional interior radius b/a can be any value between zero and one.

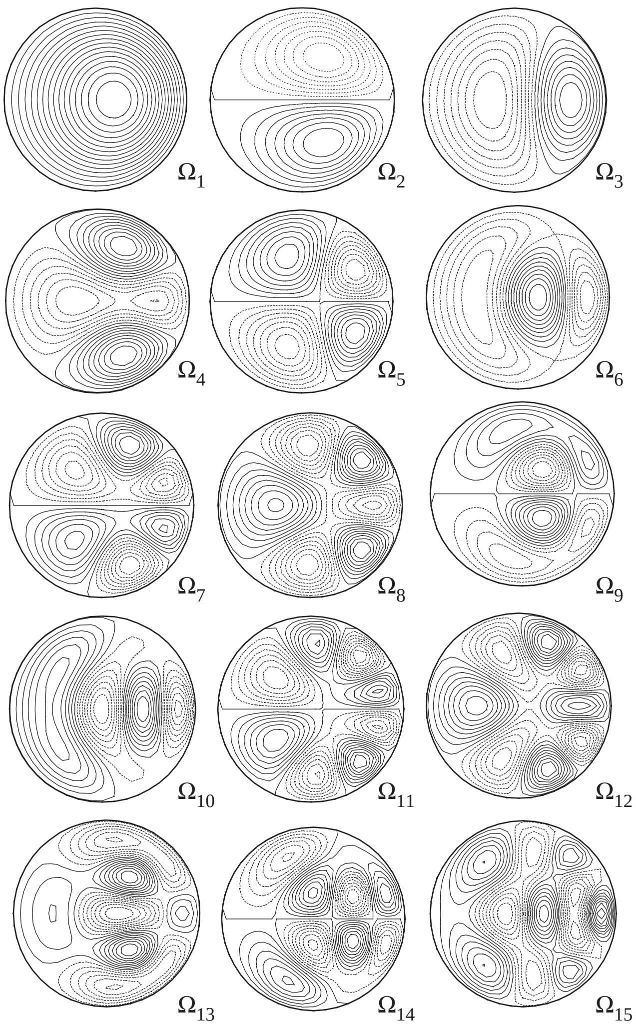

Fig. 2. Mode shapes for a variable density circular membrane with $R = 1.0$ corresponding to Table 1 with the density decreasing from left to right.

Table 2
Frequencies $\Omega = \omega a\sqrt{\rho_0/S}$ for annular membranes with variable density, $b/a = 0.2$, interior boundary fixed or free

Mode	Fixed–fixed boundaries, R						Fixed–free boundaries R					
	0.0		0.25	0.50	0.75	1.0	0.0		0.25	0.50	0.75	1.0
1	3.816	(0)	3.740	3.598	3.452	3.317	2.574	(0)	2.563	2.532	2.488	2.436
2	4.236	(1)	4.212	4.149	4.060	3.961	3.533	(1)	3.525	3.502	3.466	3.421
3	4.236	(1)	4.318	4.459	4.544	4.545	3.533	(1)	3.552	3.606	3.681	3.760
4	5.222	(2)	5.231	5.246	5.247	5.219	5.043	(2)	5.036	5.015	4.978	4.929
5	5.222	(2)	5.234	5.291	5.429	5.601	5.043	(2)	5.037	5.021	5.006	5.008
6	6.396	(3)	6.400	6.412	6.429	6.435	6.233	(0)	6.104	5.886	5.674	5.479
7	6.396	(3)	6.400	6.415	6.463	6.526	6.366	(3)	6.358	6.332	6.282	6.202
8	7.594	(4)	7.497	7.138	6.815	6.590	6.366	(3)	6.358	6.337	6.306	6.273
9	7.594	(4)	7.597	7.604	7.506	7.269	6.732	(1)	6.712	6.658	6.589	6.519
10	7.786	(0)	7.597	7.605	7.619	7.632	6.732	(1)	6.889	7.151	7.317	7.326
11	8.056	(1)	7.959	7.748	7.623	7.676	7.590	(4)	7.588	7.574	7.536	7.473
12	8.056	(1)	8.277	8.313	8.181	8.002	7.590	(4)	7.588	7.574	7.547	7.518
13	8.783	(5)	8.787	8.797	8.811	8.712	8.071	(2)	8.089	8.135	8.178	8.175
14	8.783	(5)	8.787	8.797	8.812	8.826	8.071	(2)	8.093	8.200	8.461	8.508
15	8.805	(2)	8.849	8.886	8.834	8.840	8.782	(5)	8.785	8.788	8.776	8.709

Numbers in parentheses correspond to the circular wave number for $R = 0.0$.

Table 3
Frequencies $\Omega = \omega a\sqrt{\rho_0/S}$ for annular membranes with variable density, $b/a = 0.5$, interior boundary fixed or free

Mode	Fixed–fixed boundaries, R						Fixed–free boundaries R					
	0.0		0.25	0.50	0.75	1.0	0.0		0.25	0.50	0.75	1.0
1	6.246	(0)	5.902	5.538	5.226	4.959	3.588	(0)	3.494	3.338	3.188	3.052
2	6.393	(1)	6.236	5.966	5.697	5.453	3.917	(1)	3.890	3.820	3.727	3.625
3	6.393	(1)	6.537	6.389	6.175	5.996	3.917	(1)	4.018	4.161	4.219	4.197
4	6.814	(2)	6.857	6.806	6.656	6.473	4.762	(2)	4.779	4.813	4.835	4.827
5	6.814	(2)	6.978	7.192	7.138	6.998	4.762	(2)	4.784	4.882	5.093	5.308
6	7.458	(3)	7.512	7.616	7.622	7.531	5.886	(3)	5.900	5.943	6.008	6.072
7	7.458	(3)	7.520	7.836	8.084	8.067	5.886	(3)	5.900	5.948	6.064	6.328
8	8.270	(4)	8.311	8.440	8.583	8.611	7.135	(4)	7.146	7.182	7.247	7.335
9	8.270	(4)	8.311	8.481	8.943	9.147	7.135	(4)	7.146	7.182	7.256	7.425
10	9.200	(5)	9.234	9.350	9.551	9.705	8.424	(5)	8.432	8.448	7.970	7.563
11	9.200	(5)	9.234	9.354	9.707	9.742	8.424	(5)	8.432	8.460	8.482	8.101
12	10.212	(6)	10.244	10.350	10.285	10.218	9.604	(0)	9.014	8.460	8.512	8.597
13	10.212	(6)	10.244	10.350	10.557	10.245	9.712	(6)	9.378	8.912	8.514	8.625
14	11.285	(7)	11.317	10.927	10.594	10.759	9.712	(6)	9.717	9.378	9.008	8.660
15	11.285	(7)	11.317	11.370	10.769	10.809	9.714	(1)	9.717	9.735	9.546	9.238

Numbers in parentheses correspond to the circular wave number for $R = 0.0$.

Results for frequency of vibration for a circular membrane are given in Table 1. Exact frequencies given by Rayleigh [3] are tabulated in the first column and the finite element solution agrees with the exact solution for the 15 frequencies that are tabulated. Additional results for R greater than 0.0 are tabulated. The mode shapes corresponding to $R = 1.0$ are shown in Fig. 2 and can be contrasted with the results given in Ref. [3] ($R = 0$) for a circular membrane.

Additional results for annular circular membranes with variable density are given in Tables 2 and 3. Table 2 corresponds to an annular membrane with interior radius of 0.2 and both fixed and free interior boundary condition. Numbers in parentheses correspond to the circular wave number for a membrane with constant density, $R = 0.0$. In some cases the results for the constant density membrane could be compared with previous results [4] and the agreement was excellent.

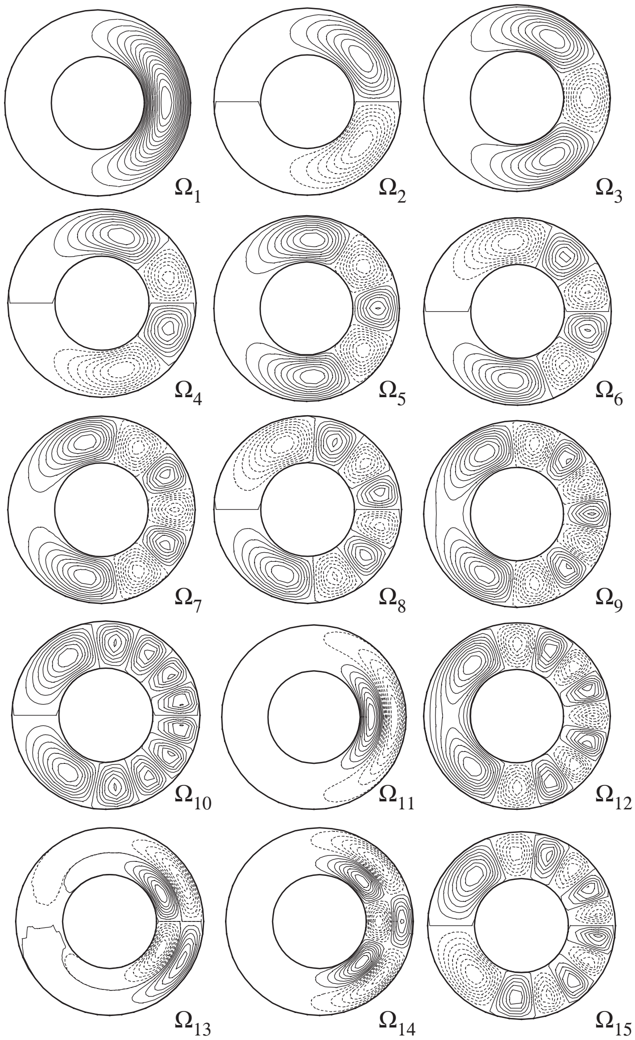

Fig. 3. Mode shapes for a variable density annular membrane with $R = 1.0$, $a/b = 0.5$ and fixed–fixed boundary conditions corresponding to Table 3 with the density decreasing from left to right.

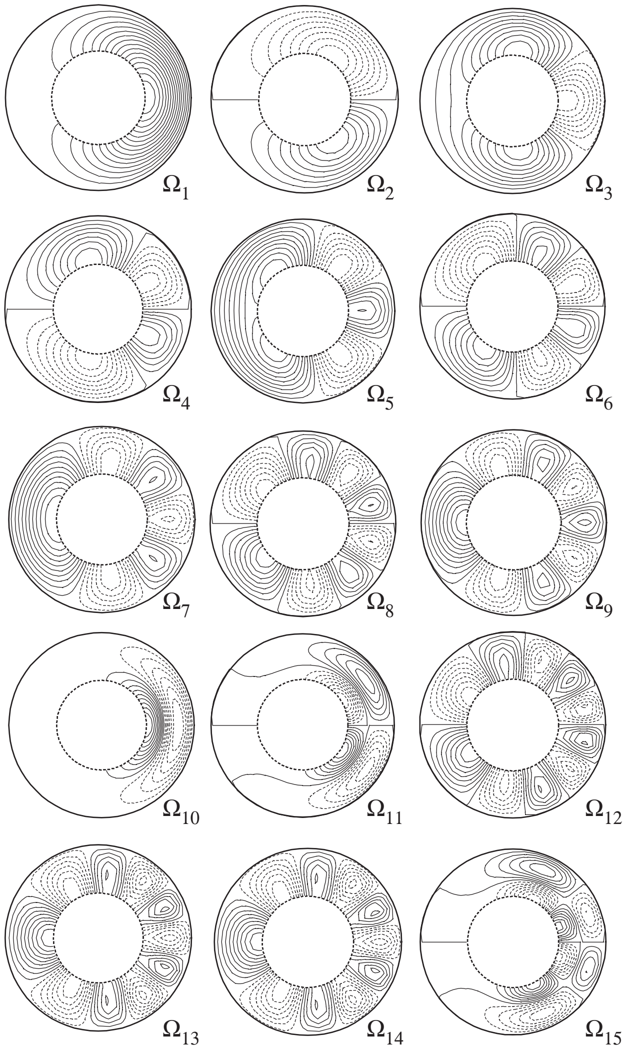

Fig. 4. Mode shapes for a variable density annular membrane with $R = 1.0$, $a/b = 0.5$ and fixed–free boundary conditions corresponding to Table 3 with the density decreasing from left to right.

Frequencies for membranes with interior radius of 0.5 are given in Table 3. The mode shapes for membranes with $R = 1.0$ are shown in Figs. 3 and 4. The mode shapes for fixed–fixed boundary conditions are compared with those for the fixed–free boundary conditions. Again the numbers in parentheses correspond to the circular wave number for a membrane with constant density.

5. Conclusions

The vibrational properties of a membrane with linear variation in density along a diameter have been studied. The mathematical problem was formulated in polar coordinates and subsequently numerical results were obtained using a finite element that was formulated in polar coordinates. Frequency of vibration was reported in tabular format and some mode shapes were presented as contour plots of the membrane deflection. It appears that results of this type have not been previously reported in the literature.

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